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Nonhomogeneous Poisson Nonlinear Transformations on Countable Infinite Set

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ABSTRACT

In this paper, we construct the family of nonhomogeneous Poisson quadratic stochastic operators defined on the countable sample space of nonnegative integers and investigate their trajectory behavior. Such operators can be reinterpreted in terms of of evolutionary operator of free population. We show that nonhomogeneous Poisson quadratic stochastic operators are regular transformations.

Keywords: Ergodic hypothesis, Poisson distribution, Quadratic stochastic operator.

1. Introduction

Let X be a countable infinite state space, $\mathcal{F}$ be σ-algebra on X, and $S(X, \mathcal{F})$ be the set of all probability measures on a measurable space $(X, \mathcal{F})$. Let $\{P(i, j, A) : i, j \in X, A \in \mathcal{F}\}$ be a family of functions on $X \times X \times \mathcal{F}$ such that $P(i, j, \cdot) \in S(X, \mathcal{F})$ and $P(i, j, A) = P(j, i, A)$ for any fixed $i, j \in X$, and $A \in \mathcal{F}$. We consider a nonlinear transformation called quadratic stochastic operator (QSO) $V : S(X, \mathcal{F}) \to S(X, \mathcal{F})$ defined by

$$(V\lambda)(A) = \int_X \int_X P(i, j, A) d\lambda(i) d\lambda(j). \quad (1)$$

where $A \in \mathcal{F}$ is an arbitrary measurable set.

If a state space $X = \{1, 2, \cdots, m\}$ be a finite set and corresponding σ-algebra $\mathcal{F}$ be a power set $\mathcal{P}(X)$, i.e., the set of all subsets of X, then the set of all probability measures on $(X, \mathcal{F})$ has the following form:

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \cdots, x_m) \in R^m : x_i \geq 0 \text{ for any } i, \text{ and } \sum_{i=1}^{m} x_i = 1\}. \quad (2)$$

that is called a $(m-1)$-dimensional simplex.

In this case, a probabilistic measure $P(i, j, \cdot)$ for any $i, j \in X$ is a discrete measure with $\sum_{k=1}^{m} P(ij, \{k\}) = 1$, where $P(ij, \{k\}) \equiv P_{ij,k}$ and corresponding QSO V has the following form

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j. \quad (3)$$

for any $\mathbf{x} \in S^{m-1}$ and for all $k = 1, \cdots, m$, where

$$a) P_{ij,k} \geq 0. \qquad b) P_{ij,k} = P_{ji,k} \text{ for all } i, j, k. \qquad c) \sum_{k=1}^{m} P_{ij,k} = 1$$

.

Such operator can be reinterpreted in terms of evolutionary operator of free population and in this form it has a fair history (see Bernstein (1924), Ganikhodjaev (1993), Ganikhodjaev (1994), Ganikhodzhaev and Eshmamatova (2006),Ganikhodzhaev et al. (2011), Jenks (1969), Kesten (1970), Losert and Akin (1983), Lyubich (1992), and Volterra (1931)).

In this paper, we construct the family of nonhomogeneous Poisson quadratic stochastic operators defined on the countable sample space of nonnegative integers and investigate their trajectory behaviors.

2. A Poisson QSO

Let $X = \{0, 1, \cdots\}$ be a countable sample space and corresponding σ-algebra $\mathcal{F}$ be a power set of X. A probability measure μ is defined on each singleton $\{k\}$, where $k = 0, 1, \cdots$ and it is written as $\mu(k)$ instead of $\mu(\{k\})$. Assume that $\{P(i, j, k) : i, j, k \in X\}$ be a family of functions defined on $X \times X \times \mathcal{F}$, that satisfy the following conditions:

i) for any fixed $i, j \in X$, $P(i, j, \cdot)$ is a probability measure on $(X, \mathcal{F})$, and
ii) for any fixed $i, j \in X$, $P(i, j, k) = P(j, i, k) \equiv P_{ij,k}$, where $k \in X$.

In this case, a QSO (1) on measurable space $(X, \mathcal{F})$ is defined as follows for arbitrary measure $\mu \in S(X, \mathcal{F})$,

$$V\mu(k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j). \tag{4}$$

where $k \in X$.

In this paper, we consider a Poisson QSO. Remind that a Poisson distribution P_λ with a positive real parameter λ is defined on X by the equation

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}. \tag{5}$$

for any $k \in X$.

Let $S(X, \mathcal{F})$ be a set of all probability measures on $(X, \mathcal{F})$ and let for any $i, j \in X, P(i, j, \cdot) \in S(X, \mathcal{F})$ be a probability measure on $(X, \mathcal{F})$.

Definition 2.1. *A quadratic stochastic operator V (4) is called a Poisson QSO if for any $i, j \in X$, the probability measure $P(i, j, \cdot)$ is the Poisson distribution $P_{\lambda(i,j)}$ with positive real parameters $\lambda(i, j)$, where $\lambda(i, j) = \lambda(j, i)$.*

Let $\Lambda = \{\lambda(i, j) : i, j \in X\}$ be a set of all possible values $\lambda(i, j)$ when i, j run the set X.

Definition 2.2. *We call a Poisson QSO V (4) homogeneous, if $|\Lambda| = 1$, i.e., $\lambda(i, j) = \lambda$, and for any $i, j \in X$, $P_{ij,k} = e^{-\lambda}\frac{\lambda^k}{k!}$.*

In this paper, we study about nonhomogeneous Poisson QSO.

3. Ergodicity and Regularity of QSO

Let us consider a QSO V (4) defined on countable set X. Assume $\{V^n\mu : n = 0, 1, 2, \cdots\}$ is the trajectory of the initial point $\mu \in S(X, \mathcal{F})$, where $V^{n+1}\mu = V(V^n\mu)$ for all $n = 0, 1, 2, \cdots$, with $V^0\mu = \mu$.

Definition 3.1. *A measure $\mu \in S(X, \mathcal{F})$ is called a fixed point of a QSO V if $V\mu = \mu$.*

Let $Fix(V)$ be the set of all fixed points of QSO V.

Definition 3.2. *A QSO V is called regular if for any initial point $\mu \in S(X, \mathcal{F})$ the limit*

$$\lim_{n\to\infty} V^n(\mu). \tag{6}$$

exists.

Proposition 3.1. *A homogeneous Poisson QSO is a regular transformation.*

Proof It is evident that for arbitrary measure $\mu \in S(X, \mathcal{F})$

$$V\mu(k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) = e^{-\lambda}\frac{\lambda^k}{k!}. \tag{7}$$

where $k \in X$, i.e., $V\mu = P_\lambda$. Thus $V^n\mu = P_\lambda$ for any $n = 1, 2, \cdots$, i.e., $Fix(V) = P_\lambda$ and

$$\lim_{n\to\infty} V^n(\mu) = P_\lambda. \tag{8}$$

In measure theory, there are various notions of the convergence of measures: weak convergence, strong convergence, total variation convergence. Below we consider strong convergence.

Definition 3.3. *For $(X, \mathcal{F})$ a measurable space, a sequence μ_n is said to converge strongly to a limit μ if*

$$\lim_{n\to\infty} \mu_n(A) = \mu(A). \tag{9}$$

for every set $A \in \mathcal{F}$.

If X is a countable set, then a sequence μ_n converges strongly to a limit μ if and only if

$$\lim_{n\to\infty} \mu_n(k) = \mu(k). \tag{10}$$

for every singleton $k \in X$.

In statistical mechanics, the ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory, the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time averages may be replaced by space averages. For nonlinear dynamical systems, (Ulam, 1960) suggested as analogue of measure-theoretic ergodicity, following ergodic hypothesis:

Definition 3.4. *A nonlinear operator V defined on $S(X,\mathcal{F})$ is called ergodic, if the limit*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}V^k\lambda. \tag{11}$$

exists for any $\lambda\in S(X,F)$.

On the basis of numerical calculations for quadratic stochastic operators defined on $S(X,F)$ with finite X, (Ulam, 1960) has conjectured that the ergodic theorem holds for any such QSO V. In 1978, (Zakharevich, 1978) has proved that this conjecture is false in general. He considered following operator on S^2

$$\begin{aligned} {x_1}' &= {x_1}^2 + 2x_1x_2,\\ {x_2}' &= {x_2}^2 + 2x_2x_3,\\ {x_3}' &= {x_3}^2 + 2x_1x_3. \end{aligned} \tag{12}$$

and proved that it is non-ergodic transformation. Later in 2004, (Ganikhodjaev and Zanin, 2004) has established sufficient condition to be non-ergodic transformation for QSO defined on S^2.

In next section, we show that Ulam's conjecture is true for some class of nonhomogeneous Poisson QSO.

4. Ergodicity and Regularity of Poisson QSO

Let V (4) be a nonhomogeneous Poisson QSO, i.e., $|\Lambda|>1$. Assume that $\Lambda=\{\lambda_0,\lambda_1,\cdots,\lambda_{m-1}\}$, i.e., $|\Lambda|=m$. It is known that the set X of all nonnegative integers forms a semigroup with operation of addition. Let $\{N_0,N_1,\cdots,N_{m-1}\}$ be a partition of the set X, where $N_s=\{n\in X: n=s(\mathrm{mod} m)\}$, with $0\le s\le m-1$. We consider the following class of nonhomogeneous Poisson QSO such that

$$P_{ij,k}=e^{-\lambda_s}\frac{\lambda_s^k}{k!}\quad\text{if}\quad i+j=s(\mathrm{mod}\ m). \tag{13}$$

For any measure $\mu\in S(X,\mathcal{F})$, let

$$A_s(\mu)=\sum_{n\in N_s}\mu(n). \tag{14}$$

where $\sum_{s=0}^{m-1} A_s(\mu) = 1$.

Then, for any initial measure $\mu \in S(X, \mathcal{F})$, we have

$$\begin{aligned} V\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) = \sum_{s=0}^{m-1}\left[\sum_{i+j=s(modm)}^{\infty} P_{ij,k}\mu(i)\mu(j)\right] \\ &= \sum_{s=0}^{m-1} e^{-\lambda_s}\frac{\lambda_s^k}{k!}\left[\sum_{i,j=0:i+j=s(modm)}^{\infty} \mu(i)\mu(j)\right] \\ &= \sum_{s=0}^{m-1} e^{-\lambda_s}\frac{\lambda_s^k}{k!}\left[\sum_{p,q=0:p+q=s(modm)}^{m-1} A_p(\mu)A_q(\mu)\right]. \end{aligned}$$

and

$$\begin{aligned} V^2\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}V\mu(i)V\mu(j) = \sum_{s=0}^{m-1}\left[\sum_{i+j=s(modm)}^{\infty} P_{ij,k}V\mu(i)V\mu(j)\right] \\ &= \sum_{s=0}^{m-1} e^{-\lambda_s}\frac{\lambda_s^k}{k!}\left[\sum_{i,j=0:i+j=s(modm)}^{\infty} V\mu(i)V\mu(j)\right] \\ &= \sum_{s=0}^{m-1} e^{-\lambda_s}\frac{\lambda_s^k}{k!}\left[\sum_{p,q=0:p+q=s(modm)}^{m-1} A_p(V\mu)A_q(V\mu)\right]. \end{aligned}$$

By simple calculations, we have

$$A_s(V\mu) = \sum_{s=0}^{m-1} A_s(P_{\lambda_k})\left[\sum_{p,q=0:p+q=k(modm)}^{m-1} A_p(\mu)A_q(\mu)\right]. \tag{15}$$

where P_{λ_k} is the Poisson distribution with parameter λ_k. Thus by induction for sequence $V^n(\mu)$ we produce the following recurrent equation

$$V^{n+1}\mu(k) = \sum_{s=0}^{m-1} e^{-\lambda_s}\frac{\lambda_s^k}{k!}\left[\sum_{p,q=0:p+q=s(modm)}^{m-1} A_p(V^n\mu)A_q(V^n\mu)\right]. \tag{16}$$

where $n = 0, 1, \cdots,$.

Besides, for parameters $\{A_s(V^n\mu) : s = 0, 1, \cdots, m-1\}$ we have the following recurrent equations

$$A_s(V^{n+1}\mu) = \sum_{s=0}^{m-1} A_s(P_{\lambda_k}) \left[\sum_{p,q=0:p+q=k(mod m)}^{m-1} A_p(V^n\mu)A_q(V^n\mu) \right]. \quad (17)$$

where $s = 0, 1, \cdots, m-1$.

It is evident that the limit behavior of the recurrent equation (16) is fully determined by limit behavior of recurrent equations (17). Since $\sum_{s=0}^{m-1} A_s(V^n\mu) = 1$ and $A_s(V^n\mu) \geq 0$ for any $s = 0, 1, \cdots, m-1$, the recurrent equations ((16)) one can consider as QSO W defined on $m-1$ dimensional simplex S^{m-1}.

Thus, the problem of investigating QSO defined on countable state space is reduced to problem of investigation the QSO defined on finite state space. It is evident that the nonhomogeneous QSO be a regular transformations if and only if the QSO W is regular transformation.

In (Ganikhodjaev and Hamzah, 2014), the authors proved that nonhomogeneous Poisson QSO with $m = 2$ and $m = 3$ are regular transformations. Using numerical analysis one can show that the QSO W is regular transformation for any m. Below we consider the cases $m = 4$ and $m = 5$.

4.1 Nonhomogeneous Poisson QSO with $m = 4$

For $m = 4$, using simple calculations, one can show that

$$\begin{aligned} A_0(\lambda) &= \tfrac{1+e^{-2\lambda}+2e^{-\lambda}\cos(\lambda)}{4}, \quad A_1(\lambda) = \tfrac{1-e^{-2\lambda}-2e^{-\lambda}\sin(\lambda)}{4}, \\ A_2(\lambda) &= \tfrac{1+e^{-2\lambda}-2e^{-\lambda}\cos(\lambda)}{4}, \quad A_3(\lambda) = \tfrac{1-e^{-2\lambda}+2e^{-\lambda}\sin(\lambda)}{4}. \end{aligned} \quad (18)$$

$$\begin{aligned} V^{n+1}\mu(k) &= e^{-\lambda_0}\tfrac{\lambda_0^k}{k!}\left[A_0^2(V^n\mu) + 2A_1(V^n\mu)A_3(V^n\mu) + A_2^2(V^n\mu)\right] \\ &+ e^{-\lambda_1}\tfrac{\lambda_1^k}{k!}\left[2A_0(V^n\mu)A_1(V^n\mu) + 2A_2(V^n\mu)A_3(V^n\mu)\right] \\ &+ e^{-\lambda_2}\tfrac{\lambda_2^k}{k!}\left[A_1^2(V^n\mu) + 2A_0(V^n\mu)A_2(V^n\mu) + A_3^2(V^n\mu)\right] \\ &+ e^{-\lambda_3}\tfrac{\lambda_3^k}{k!}\left[2A_0(V^n\mu)A_3(V^n\mu) + 2A_1(V^n\mu)A_2(V^n\mu)\right]. \end{aligned} \quad (19)$$

and

$$\begin{aligned} A_s(V^{n+1}\mu) &= A_s(\lambda_0)\left[A_0^2(V^n\mu)+2A_1(V^n\mu)A_3(V^n\mu)+A_2^2(V^n\mu)\right] \\ &+ A_s(\lambda_1)\left[2A_0(V^n\mu)A_1(V^n\mu)+2A_2(V^n\mu)A_3(V^n\mu)\right] \\ &+ A_s(\lambda_2)\left[A_1^2(V^n\mu)+2A_0(V^n\mu)A_2(V^n\mu)+A_3^2(V^n\mu)\right] \\ &+ A_s(\lambda_3)\left[2A_0(V^n\mu)A_3(V^n\mu)+2A_1(V^n\mu)A_2(V^n\mu)\right]. \end{aligned} \tag{20}$$

where $s = 0, 1, 2, 3$.

Starting from arbitrary initial data we iterate the recurrence equations (20) and observe their behavior after a large number of iterations. The resultant diagrams in the space (λ_0, λ_1) with $0 < \lambda_0, \lambda_1 \leq 2$ and some fixed λ_2 and λ_3, are shown in figure below.

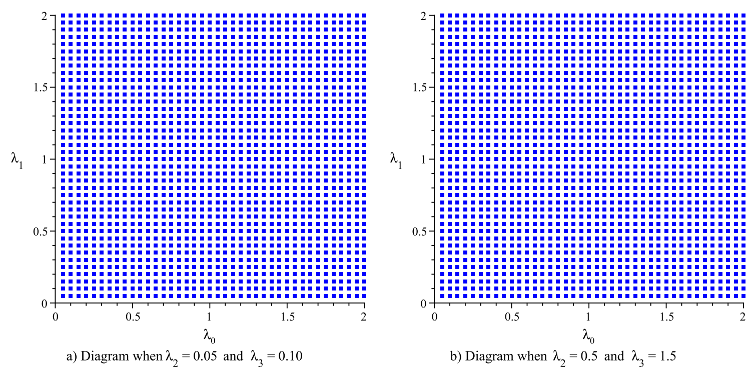


Figure 1: Limit behavior of the dynamical system (20) $0 < \lambda_0, \lambda_1 \leq 2$ and some fixed values λ_2 and λ_3.

In this Figure 1, blue color corresponds to converges of the trajectory. Note that if these parameters are very small, then any trajectory converges to $(1, 0, 0, 0)$, while if they are too large, then any trajectory converges to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

Thus, for any initial measure μ, we have

$$\lim_{n\to\infty} A_s(V^n\mu) = x_s^*. \tag{21}$$

with $s = 0, 1, 2, 3$. Then passing to limit in (19), we have that for any singleton

k

$$\lim_{n\to\infty} V^{n+1}\mu(k) = e^{-\lambda_0}\frac{\lambda_0^k}{k!}\left[{x_0^*}^2 + 2x_1^*x_3^* + {x_2^*}^2\right] + e^{-\lambda_1}\frac{\lambda_1^k}{k!}\left[2x_0^*x_1^* + 2x_2^*x_3^*\right]$$
$$+ e^{-\lambda_2}\frac{\lambda_2^k}{k!}\left[{x_1^*}^2 + 2x_0^*x_2^* + {x_3^*}^2\right] + e^{-\lambda_3}\frac{\lambda_3^k}{k!}\left[2x_0^*x_3^* + 2x_1^*x_2^*\right].$$

Thus, for any initial measure μ, the strong limit of the sequence $V^n\mu$ is exists and equal to the convex linear combination

$$\begin{aligned}\lim_{n\to\infty} V^n\mu &= \left[{x_0^*}^2 + 2x_1^*x_3^* + {x_2^*}^2\right]P(\lambda_0) + \left[2x_0^*x_1^* + 2x_2^*x_3^*\right]P(\lambda_1)\\ &+ \left[{x_1^*}^2 + 2x_0^*x_2^* + {x_3^*}^2\right]P(\lambda_2) + \left[2x_0^*x_3^* + 2x_1^*x_2^*\right]P(\lambda_3).\end{aligned} \tag{22}$$

of the four Poisson measures $\{P_{\lambda_s} : s = 0, 1, 2, 3\}$. As corollary we have the following statement:

Proposition 4.1. *Nonhomogeneous Poisson QSO with four different parameters is a regular and respectively ergodic transformation with respect to strong convergence.*

4.2 Nonhomogeneous Poisson QSO with $m = 5$

For $m = 5$, using simple but tedious calculations, one can show that

$$A_0(\lambda) = \frac{2}{5}\left[e^{-\lambda+\lambda\cos(\frac{2\pi}{5})}\cos\left(\lambda\sin\left(\frac{2\pi}{5}\right)\right) + e^{-\lambda-\lambda\cos(\frac{\pi}{5})}\cos\left(\lambda\sin\left(\frac{\pi}{5}\right)\right) + \frac{1}{2}\right],$$

$$A_1(\lambda) = \frac{1}{5[\cos\left(\frac{2\pi}{5}\right) + \cos(\frac{\pi}{5})]}\{\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + e^{-\lambda-\lambda\cos(\frac{\pi}{5})}$$
$$\left[\sin\left(\frac{\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} - \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) - \cos\left(\lambda\sin\left(\frac{\pi}{5}\right)\right)\right]$$
$$+ \quad e^{-\lambda+\lambda\cos(\frac{\pi}{5})}$$
$$\left[\sin\left(\frac{3\pi}{10} + \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) + \cos\left(\lambda\sin\right.\right.$$

$$A_2(\lambda) = \frac{1}{5[\cos(\frac{2\pi}{5}) + \cos(\frac{\pi}{5})]}\{\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + e^{-\lambda-\lambda\cos(\frac{\pi}{5})}$$
$$\left[\sin\left(\frac{3\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} - \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) + \cos\left(\lambda\sin\left(\frac{\pi}{5}\right)\right)\right.$$
$$+ \quad e^{-\lambda+\lambda\cos(\frac{2\pi}{5})}$$
$$\left[\sin\left(\frac{\pi}{10} + \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \cos\left(\lambda\sin\right.\right.$$

$$A_3(\lambda) = \frac{1}{5[\cos(\frac{2\pi}{5}) + \cos(\frac{\pi}{5})]}\{\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + e^{-\lambda-\lambda\cos(\frac{\pi}{5})}$$
$$\left[\sin\left(\frac{3\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} - \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) - \cos\left(\lambda\sin\left(\frac{\pi}{5}\right)\right)\right.$$
$$+ \quad e^{-\lambda+\lambda\cos(\frac{\pi}{5})}$$
$$\left[\sin\left(\frac{3\pi}{10} + \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) + \cos\left(\lambda\sin\right.\right.$$

$$A_4(\lambda) = \frac{1}{5[\cos(\frac{2\pi}{5}) + \cos(\frac{\pi}{5})]}\{\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + e^{-\lambda-\lambda\cos(\frac{\pi}{5})}$$
$$\left[\sin\left(\frac{\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} - \lambda\sin\left(\frac{\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} + \lambda\sin\left(\frac{\pi}{5}\right)\right) - \cos\left(\lambda\sin\left(\frac{\pi}{5}\right)\right)\right]$$
$$+ \quad e^{-\lambda+\lambda\cos(\frac{\pi}{5})}$$
$$\left[\sin\left(\frac{3\pi}{10} + \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{3\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) - \sin\left(\frac{\pi}{10} - \lambda\sin\left(\frac{2\pi}{5}\right)\right) + \cos\left(\lambda\sin\right.\right.$$

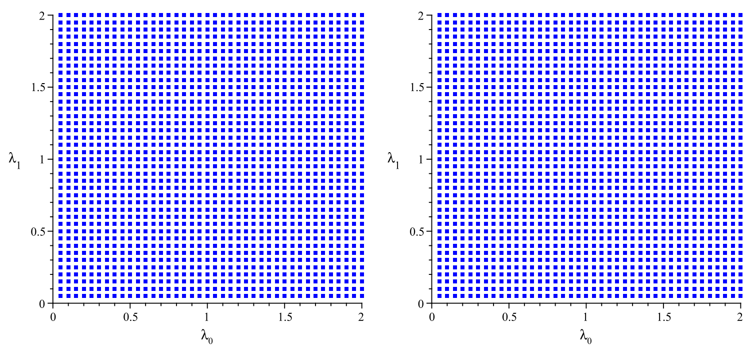
$$\begin{aligned}
V^{n+1}\mu(k) &= e^{-\lambda_0}\frac{\lambda_0^k}{k!}\left[A_0^2(V^n\mu) + 2A_1(V^n\mu)A_4(V^n\mu) + 2A_2(V^n\mu)A_3(V^n\mu)\right] \\
&+ e^{-\lambda_1}\frac{\lambda_1^k}{k!}\left[A_3^2(V^n\mu) + 2A_0(V^n\mu)A_1(V^n\mu) + 2A_2(V^n\mu)A_4(V^n\mu)\right] \\
&+ e^{-\lambda_2}\frac{\lambda_2^k}{k!}\left[A_1^2(V^n\mu) + 2A_0(V^n\mu)A_2(V^n\mu) + 2A_2(V^n\mu)A_4(V^n\mu)\right] \\
&+ e^{-\lambda_3}\frac{\lambda_2^k}{k!}\left[A_4^2(V^n\mu) + 2A_0(V^n\mu)A_3(V^n\mu) + 2A_1(V^n\mu)A_2(V^n\mu)\right] \\
&+ e^{-\lambda_4}\frac{\lambda_3^k}{k!}\left[A_2^2(V^n\mu) + 2A_0(V^n\mu)A_4(V^n\mu) + 2A_1(V^n\mu)A_3(V^n\mu)\right]
\end{aligned} \tag{23}$$

and

$$\begin{aligned} A_s(V^{n+1}\mu) &= A_s(\lambda_0)\left[A_0^2(V^n\mu)+2A_1(V^n\mu)A_4(V^n\mu)+2A_2(V^n\mu)A_3(V^n\mu)\right] \\ &+ A_s(\lambda_1)\left[A_3^2(V^n\mu)+2A_0(V^n\mu)A_1(V^n\mu)+2A_2(V^n\mu)A_4(V^n\mu)\right] \\ &+ A_s(\lambda_2)\left[A_1^2(V^n\mu)+2A_0(V^n\mu)A_2(V^n\mu)+2A_2(V^n\mu)A_4(V^n\mu)\right] \\ &+ A_s(\lambda_3)\left[A_4^2(V^n\mu)+2A_0(V^n\mu)A_3(V^n\mu)+2A_1(V^n\mu)A_2(V^n\mu)\right] \\ &+ A_s(\lambda_4)\left[A_2^2(V^n\mu)+2A_0(V^n\mu)A_4(V^n\mu)+2A_1(V^n\mu)A_3(V^n\mu)\right] \end{aligned} \tag{24}$$

where $s = 0, 1, 2, 3, 4$.

Starting from arbitrary initial data we iterate the recurrence equations (24) and observe their behavior after a large number of iterations. The resultant diagrams in the space (λ_0, λ_1) with $0 < \lambda_0, \lambda_1 \leq 2$ and some fixed λ_2, λ_3 and λ_4, are shown in figure below.



a) Diagram when $\lambda_2 = 0.25$, $\lambda_3 = 0.75$ and $\lambda_4 = 1.8$

b) Diagram when $\lambda_2 = 1.55$, $\lambda_3 = 0.45$ and $\lambda_4 = 0.35$

Figure 2: Limit behavior of the dynamical system (24) $0 < \lambda_0, \lambda_1 \leq 2$ and some fixed values λ_2, λ_3, and λ_4.

In this Figure 2, blue color corresponds to converges of the trajectory. Note that if these parameters are very small, then any trajectory converges to $(1, 0, 0, 0, 0)$, while if they are too large, then any trajectory converges to $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$

Thus we have proved the following statement.

Proposition 4.2. *A nonhomogeneous Poisson QSO with five different parameters is a regular and respectively ergodic transformation with respect to strong convergence.*

Follow this line one can prove the regularity of nonhomogeneous Poisson QSO for any m.

5. Conclusion

In this article, we present a construction of nonhomogeneous Poisson quadratic stochastic operators with any finitely many different parameters λ_i and prove regularity of these operators.

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